INTRODUCTION TO
DIFFERENTIAL EQUATIONS

Dr. Kalipada Maity

Associate Professor
Department of Mathematics
Mugberia Gangadhar Mahavidyalaya, Bhupatinagar
Purba Medinipur-721425, West Bengal, India

To my beloved Daughters

Samadrita & Somdatta

Preface

With the remarkable advancement in various branches of science, engineering and technology, today more than ever before, the study of differential equations has become essential. For, to have an exhaustive understanding of subjects like physics, mathematical biology, chemical science, mechanics, fluid dynamics, heat transfer, aerodynamics, electricity, waves and electromagnetic, the knowledge of finding solution to differential equations is absolutely necessary. These differential equations may be ordinary or partial. Finding and interpreting their solutions are at the heart of applied mathematics. A thorough introduction to differential equations is therefore a necessary part of the education of any applied mathematician, and this book is aimed at building up skills in this area.

This book on ordinary / partial differential equations is the outcome of a series of lectures delivered by me, over several years, to the undergraduate or postgraduate students of Mathematics at various institution. My principal objective of the book is to present the material in such a way that would immediately make sense to a beginning student. In this respect, the book is written to acquaint the reader in a logical order with various well-known mathematical techniques in differential equations. Besides, interesting examples solving JAM / GATE / NET / IAS / NBHM/TIFR/SSC questions are provided in almost every chapter which strongly stimulate and help the students for their preparation of those examinations from graduate level.

Organization of the book
The book has been organized in a logical order and the topics are discussed in a systematic manner. It has comprising 21 chapters altogether. In the chapter **??**, the fundamental concept of differential equations including autonomous/ non-autonomous and linear / non-linear differential equations has been explained. The order and degree of the ordinary differential equations (ODEs) and partial differential equations(PDEs) are also mentioned. The chapter **??** are concerned the first order and first degree ODEs. It is also written in a progressive manner, with the aim of developing a deeper understanding of ordinary differential equations, including conditions for the existence and uniqueness of solutions. In chapter **??** the first order and higher degree ODEs are illustrated with sufficient examples. The chapter **??** is concerned with the higher order and first degree ODEs. Several methods, like method of undetermined coefficients, variation of parameters and Cauchy-Euler equations are also introduced in this chapter. In chapter 1, second order initial value problems, boundary value problems and Eigenvalue problems with Sturm-Liouville problems are expressed with proper examples. Simultaneous linear differential equations are studied in chapter **??**. It is also written in a progressive manner with the aim of developing some alternative methods. In chapter **??**, the equilibria, stability

and phase plots of linear / nonlinear differential equations are also illustrated by including numerical solutions and graphs produced using Mathematica version 9 in a progressive manner. The geometric and physical application of ODEs are illustrated in chapter **??**. The chapter **??** is presented the Total (Pfaffian) Differential Equations. In chapter **??**, numerical solutions of differential equations are added with proper examples. Further, I discuss Fourier transform in chapter **??**, Laplace transformation in chapter **??**, Inverse Laplace transformation in chapter **??**. Moreover, series solution techniques of ODEs are presented with Frobenius method in chapter **??**, Legendre function and Rodrigue formula in Chapter **??**, Chebyshev functions in chapter **??**, Bessel functions in chapter **??** and more special functions for Hypergeometric, Hermite and Laguerre in chapter **??** in detail.

Besides, the partial differential equations are presented in chapter **??**. In the said chapter, the classification of linear, second order partial differential equations emphasizing the reasons why the canonical examples of elliptic, parabolic and hyperbolic equations, namely Laplace's equation, the diffusion equation and the wave equation have the properties that they do has been discussed. Chapter **??** is concerned with Green's function. In chapter **??**, the application of differential equations are developed in a progressive manner. Also all chapters are concerned with sufficient examples. In addition, there is also a set of exercises at the end of each chapter to reinforce the skills of the students.

Moreover it gives the author great pleasure to inform the reader that the **second edition** of the book has been improved, well -organized, enlarged and made up-to-date as per latest UGC - CBSC syllabus. The following significant changes have been made in the second edition:

- Almost all the chapters have been rewritten in such a way that the reader will not find any difficulty in understanding the subject matter.
- Errors , omissions and logical mistakes of the previous edition have been corrected.
- The exercises of all chapters of the previous edition have been improved, enlarged and well-organized.
- Two new chapters like Green's Functions and Application of Differential Equations have been added in the present edition.
- More solved examples have been added so that the reader may gain confidence in the techniques of solving problems.
- References to the latest papers of various university, IIT-JAM, GATE, and CSIR-UGC(NET) have been provided in almost every chapters which strongly help the students for their preparation of those examinations from graduate label.

In view of the above mentioned features it is expected that this new edition will appreciate and be well prepared to use the wonderful subject of differential equations.

Aim and Scope
When mathematical modelling is used to describe physical, biological or chemical phenomena, one of the most common results of the modelling process is a system of ordinary or partial differential equations. Finding and interpreting the solutions of these differential equations

is therefore a central part of applied mathematics, Physics and a thorough understanding of differential equations is essential for any applied mathematician and physicist. The aim of this book is to develop the required skills on the part of the reader. The book will thus appeal to undergraduates/postgraduates in Mathematics, but would also be of use to physicists and engineers. There are many worked examples based on interesting real-world problems. A large selection of examples / exercises including JAM/NET/GATE questions is provided to strongly stimulate and help the students for their preparation of those examinations from graduate level. The coverage is broad, ranging from basic ODE , PDE to second order ODE's including Bifurcation theory, Sturm-Liouville theory, Fourier Transformation, Laplace Transformation, Green's function and existence and uniqueness theory, through to techniques for nonlinear differential equations including stability methods. Therefore, it may be used in research organization or scientific lab.

Significant features of the book

- A complete course of differential Equations
- Perfect for self-study and class room
- Useful for beginners as well as experts
- More than 650 worked out examples
- Large number of exercises
- More than 700 multiple choice questions with answers
- Suitable for New UGC-CBSC syllabus of ODE & PDE
- Suitable for GATE, NET, NBHM, TIFR, JAM, JEST, IAS, SSC examinations.

ACKNOWLEDGEMENTS

This book is the outcome of a series of lectures and research experience carried out by me over several years. However it would not be possible to incorporate or framing the entire book without the help of many academicians. As such, I am indebted to many of my teachers and students. Especially I would like to thank Dr. Swapan Kumar Misra, Principal, Mugberia Gangadhar Mahavidyalaya for his generous support in writing this book.

I express sincerest respect to my research guide Prof. M. Maiti, Department of Mathematics, Vidyasagar University, Midnapore 721102, West Bengal, India.

I also express sincerest gratitude to my collogues: Dr. Nabakumar Ghosh, Dr. Arpan Dhara, Prof. Bikash Panda, Prof. Hiranmay Manna, Prof. Madhumita Sahoo for their positive suggestions to improve the standard of the book. Especial thanks are owed to my collogues, Dept. of Mathematics: Prof. Suman Giri, Prof. Asim Jana, Prof. Tanushri Maity, Prof. Debraj Manna for their excellent typing to add some chapters of the book. Also I wish to thank several persons of our institution who have made many encouraging remarks and constructive suggestions on the manuscript.

I would like to express sincere appreciation to my friends: Dr. Dipak Kr. Jana and Dr. Shibsankar Das for their constant source of inspiration.

My sincere appreciation goes to my students who give me a stage where I can cultivate my talent and passion for teaching. My graduate and post graduate students who have used the draft of this book as a textbook have made many encouraging comments and constructive suggestions. Also, I heartily thanks my scholars: Dr. Samar Hazari, Mr. Jatin Nath Roul, Mr. Anupam De and Mr. Debnarayan Khatua for their help in different direction to modify the book.

Without the unfailing love and support of my parents, who have always believed in me, this work would not have been possible. In addition, the care, love, patience, and understanding of my wife and lovely daughters have been of inestimable encouragement and help. I love them very much and appreciate all that they have contributed to my work.

In written this book, I have taken some helps from several books and research papers which are immensely helpful and are given in an alphabetical order in the bibliography. So I also express my gratitude and respect to all the eminent authors. I apologize if I inadvertently missed acknowledging mygratitude to any one else.

I shall feel great to receive constructive criticisms through email for the improvement of the book from the experts as well as the learners.

I thank the Narosa Publishing House Pvt. Ltd. for their sincere care in the publication of this book.

Kalipada Maity,
E-mail: kalipada_maity@yahoo.co.in / kmaity78@gmail.com

Contents

Chapter 1

Second Order Initial-value, Boundary-value and Eigenvalue Problems

1.1 Introduction

In this chapter, we consider second order differential equation along with subsidiary conditions to be satisfied by the solution of the differential equation. We discuss three types of problems known as

1. Initial-value problems,
2. Boundary-value problems and
3. Eigenvalue problems.

1.2 Initial-value problems

An initial-value problem is a differential equation together with subsidiary conditions to be satisfied by the solution function and its derivatives, all given at the same value of the independent variable. A second order initial-value problem may in general be put in the standard form as:

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = X \text{ Where } P, Q \text{ and } X \text{ are functions of } x, \text{ with the condition.} \tag{1.1}$$

$$y(a) = c_1 \text{ and } y'(a) = c_2, \tag{1.2}$$

where a is a specific value of the independent variable x and c_1, c_2 are two constants. Hence a solution to an initial-value problems is to find a $y(x)$ that satisfies the differential equation (1.1) as well as the given initial condition (1.2).

If particular, $X = 0$ and $c_1 = c_2 = 0$, the problem is said to be a homogeneous initial-value problem.

Theorem 1.1 (Existence Theorem) Let P, Q be ant two continuous functions on $[a, b]$. For any real x_0, and constants α, β there exists a solution ϕ of the initial value problem

$$y'' + P(x)y' + Q(x)y = 0, \ y(x_0) = \alpha, \ y'(x_0) = \beta \qquad \text{on [a, b]}.$$

Proof.: The proof is entirely similar to the proof of Theorem **??** of Chapter **??**.

Theorem 1.2 Let P, Q be two continuous function on $[a, b]$ and let ϕ be any solution of the equation

$$y'' + Py' + Qy = 0, \ y(x_0) = \alpha, \ y'(x_0) = \beta$$

on $[a, b]$ containing a point x_0. Then for all x in $[a, b]$

$$||\phi(x_0)||e^{-k|x-x_0|} \leq ||\phi(x)|| \leq ||\phi(x_0)||e^{k|x-x_0|}, \text{ where } ||\phi(x)|| = \sqrt{|\phi(x)|^2 + |\phi'(x)|^2}, \, k = 1 + |P| + |Q|.$$

Proof.: The proof is entirely similar to the proof of Theorem **??** of Chapter **??**.

Theorem 1.3 (Uniqueness Theorem) Let α, β be any two constants and let x_0 be any real number. On any interval $[a, b]$ containing x_0 there exists at most one solution ϕ of the initial value problem

$$y'' + P(x)y' + Q(x)y = 0, \ y(x_0) = \alpha, \ y'(x_0) = \beta, \text{ where } P, Q \text{ are continuous functions on } [a, b].$$

Proof.: The proof is entirely similar to the proof of Theorem **??** of Chapter **??**.

Example 1.1 Solve: $\frac{d^2y}{dx^2} + 4y = 0; \ y(0) = 0, \ y'(0) = 2.$

Solution: The given homogeneous differential equation is

$$\frac{d^2y}{dx^2} + 4y = 0 \tag{1.3}$$

Let $y(x) = e^{mx}$ (m being a constant) be a trial solution of (1.3). Then the A.E. of (1.3) is $m^2 + 4 = 0$ or $m = \pm 2i$. So the general solution is

$$y = c_1 \cos 2x + c_2 \sin 2x \tag{1.4}$$

c_1 and c_2 being two arbitrary constants. Now, $y(0) = 0$ gives, $c_1 = 0$. Again, $y'(x) = -2c_1 \sin 2x + 2c_2 \cos 2x$. So, $y'(0) = 2$ gives, $c_2 = 1$. So the solution of initial-value problem (1.3) is $y = \sin 2x$.

1.3 Boundary-value problems

A boundary value problem in one dimension is an ordinary differential equation together with conditions involving values of the solution and/or its derivatives at two or more points. The number of conditions imposed is equal to the order of the differential equation. Usually, boundary value problems of any physical relevance have these characteristics:
(1) The conditions are imposed at two different points,
(2) the solution is of interest only between those two points,
(3) and the independent variable is a space variable, which we shall represent as x.

In addition, we are primarily concerned with cases where the differential equation is linear and of second order. However, problems in elasticity often involve fourth-order equations.

In contrast to initial value problems, even the most innocent looking boundary value problem may have exactly one solution, no solution, or an infinite number of solutions.

When the differential equation in a boundary value problem has a known general solution, we use the two boundary conditions to supply two equations that are to be satisfied by the two constants in the general solution. If the differential equation is linear, these are two linear equations and can be easily solved, if there is a solution.

A second order boundary-value problem in standard form may in general be put in the form:

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R(x), \ a < x < b \tag{1.5}$$

with the boundary conditions

$$A_1 y(a) + B_1 y'(a) = c_1, \quad A_2 y(b) + B_2 y'(b) = c_2 \tag{1.6}$$

where P, Q and R are functions of x on $[a, b]$ and $A_1, B_1, c_1, A_2, B_2, c_2$ are all real constants. Also assume that $a \neq b$, A_1 and B_1 are not zero at a time and similarly A_2 and B_2 are also not all zero at a time.

If the differential equation as well as the boundary conditions are all homogeneous, that is ,if $R(x) = 0$ on $[a, b]$ and $c_1 = c_2 = 0$, then this problem is said to be a homogeneous boundary-value problems. Thus a homogeneous boundary value problem is of the form

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0, \ a < x < b \tag{1.7}$$

with the the boundary conditions

$$A_1 y(a) + B_1 y'(a) = 0, \quad A_2 y(b) + B_2 y'(b) = 0 \tag{1.8}$$

Hence, a solution to a non-homogeneous boundary value problem is to find a $y(x)$ that satisfies the differential equation (1.5) as well as the given boundary condition (1.6). Also a solution to a homogeneous boundary value problem is to find a $y(x)$ that satisfies the differential equation (1.7) as well as the given boundary condition (1.8). The problem (1.11) with the condition (1.12) always have the trivial solution $y(x) = 0$.

Example 1.2 Solve: $\frac{d^2y}{dx^2} + 4y = 0; \ 0 < x < \frac{\pi}{4}$, with $y(0) = 1, \ y(\frac{\pi}{4}) = 2$.

Solution: The given homogeneous differential equation is

$$\frac{d^2y}{dx^2} + 4y = 0, \ 0 < x < \frac{\pi}{4} \tag{1.9}$$

Let $y(x) = e^{mx}$ (m being a constant) be a trial solution of (1.9). Then the A.E. of (1.9) is $m^2 + 4 = 0$ or $m = \pm 2i$. So the general solution is

$$y = c_1 \cos 2x + c_2 \sin 2x \tag{1.10}$$

where c_1 and c_2 being two arbitrary constants. Now, $y(0) = 1$ gives $c_1 = 1$ and $y(\frac{\pi}{4}) = 2$ gives $c_2 = 2$. So the solution of the boundary value problem (1.9) is $y = \cos 2x + 2\sin 2x$ on $[0, \frac{\pi}{4}]$.

1.3.1 More General Type of Homogeneous Boundary-value Problems

A more general type of homogeneous boundary-value problem is one in which the co-efficient $P(x)$ and $Q(x)$ are also depend on an arbitrary constants λ on $[a, b]$. This problem has the form:

$$\frac{d^2y}{dx^2} + P(x,\lambda)\frac{dy}{dx} + Q(x,\lambda)y = 0, \ a < x < b \tag{1.11}$$

with the conditions

$$A_1 y(a) + B_1 y'(a) = 0, \qquad A_2 y(b) + B_2 y'(b) = 0 \tag{1.12}$$

It is obvious that one trivial solution of (1.11) with the condition (1.12) is $y(x) = 0, \ a \le x \le b$.

Theorem 1.4 Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0, \ a < x < b$$

with the boundary conditions

$$A_1 y(a) + B_1 y'(a) = 0, \ \ A_2 y(b) + B_2 y'(b) = 0$$

Then non-trivial solution of this problem will exit if and only if

$$\Delta = \begin{vmatrix} A_1 y_1(a) + B_1 y_1'(a) & A_1 y_2(a) + B_1 y_2'(a) \\ A_2 y_1(b) + B_2 y_1'(b) & A_2 y_2(b) + B_2 y_2'(b). \end{vmatrix} = 0 \tag{1.13}$$

Theorem 1.5 A non homogeneous problem has a unique solution if and only if the associated homogeneous problem has a unique solution, i.e the homogeneous problem has only the trivial solution.

Example 1.3 Solve $y'' = 0, \ -1 < x < 1$ subject to $y(-1) = 0, \ y(1) - 2y'(1) = 0$.

Solution: The given homogeneous differential equation is

$$y'' = 0 \tag{1.14}$$

Let $y(x) = e^{mx}$ (m being a constant) be a trial solution of (1.14). Then the A.E. of (1.14) will be $m^2 = 0$ or $m = 0, 0$. So the general solution is

$$y(x) = c_1 + c_2 x, \ -1 \le x \le 1 \tag{1.15}$$

where c_1, c_2 are arbitrary constants. Here the boundary conditions are $y(-1) = 0, \ y(1) - 2y'(1) = 0$. It is also a boundary value problem. The existence of non-trivial solutions is also immediate from Theorem 1.4. Here $y_1(x) = 1$ and $y_2(x) = x$. Then the determinant

$$\begin{vmatrix} A_1 y_1(a) + B_1 y_1'(a) & A_1 y_2(a) + B_1 y_2'(a) \\ A_2 y_1(b) + B_2 y_1'(b) & A_2 y_2(b) + B_2 y_2'(b) \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix} = 0 \tag{1.16}$$

where $A_1 = 1$, $B_1 = 0$, $A_2 = 1$, $B_2 = -2$, $a = -1$ and $b = 1$. So the problem has non-trivial solution. Using the given conditions in (1.15) we get, $c_1 = c_2$ where c_2 is an arbitrary constant.

Hence, the solution to the given boundary value problem is $y = c_2(1 + x)$, $-1 \le x \le 1$ where c_2 is an arbitrary constant. For different values of c_2, the problem has infinite many non-trivial solutions.

Example 1.4 Solve: $y'' - 2y' + 2y = 0, 0 < x < \pi$ subject to (i) $y(0) = 0,\ y(\pi) = 0$ (ii) $y(0) = 0,\ y(\frac{\pi}{2}) = 0$.

Solution: The given homogeneous differential equation is

$$y'' - 2y' + 2y = 0 \tag{1.17}$$

Let $y(x) = e^{mx}$ (m being a constant) be a trial solution of (1.17). Then the A.E. of (1.17) will be $m^2 - 2m + 2 = 0$ or $m = 1 \pm i$. So the general solution is

$$y(x) = e^x(c_1 \cos x + c_2 \sin x),\ 0 \le x \le \pi, \tag{1.18}$$

where $c_1,\ c_2$ are arbitrary constants.

Case I: When $y(0) = 0,\ y(\pi) = 0$

It is also a boundary values problem. The existence of non-trivial solutions is also immediate from Theorem 1.4. Here $y_1(x) = e^x \cos x$ and $y_2(x) = e^x \sin x$. Then the determinant

$$\begin{vmatrix} A_1y_1(a) + B_1y_1'(a) & A_1y_2(a) + B_1y_2'(a) \\ A_2y_1(b) + B_2y_1'(b) & A_2y_2(b) + B_2y_2'(b) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & 0 \end{vmatrix} = 0 \tag{1.19}$$

where $A_1 = 1, B_1 = 0, A_2 = 1, B_2 = 0, a = 0$ and $b = \pi$. So the problem has non-trivial solution. Using the given conditions in (1.18) we get, $c_1 = 0$ but no value of c_2. Hence, the solution is $y = c_2e^x \sin x$ and this solution is valid for any values of c_2. So in this case there are infinite solutions.

Case II: When $y(0) = 0,\ y(\frac{\pi}{2}) = 0$

It is also a boundary values problem. The existence of the unique trivial solutions is also immediate from Theorem 1.4. Here $y_1(x) = e^x \cos x$ and $y_2(x) = e^x \sin x$. Then the determinant

$$\begin{vmatrix} A_1y_1(a) + B_1y_1'(a) & A_1y_2(a) + B_1y_2'(a) \\ A_2y_1(b) + B_2y_1'(b) & A_2y_2(b) + B_2y_2'(b) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & e^{\frac{\pi}{2}} \end{vmatrix} = e^{\frac{\pi}{2}} (\ne 0) \tag{1.20}$$

where $A_1 = 1, B_1 = 0, A_2 = 1, B_2 = 0, a = 0$ and $b = \frac{\pi}{2}$. So the problem has trivial solution. Using the given conditions in (1.18) we get , $c_1 = 0,\ c_2 = 0$. Hence, the trivial solution is $y(x) = 0$. So in this case there are unique solution $y(x) = 0$ on $[0, \pi]$.

Example 1.5 Solve: $y'' - 2y' + 2y = 2, 0 < x < \pi$ subject to (i) $y(0) = 1,\ y(\pi) = -1$ (ii) $y(0) = 1,\ y(\frac{\pi}{2}) = 1$.

Solution: The given non-homogeneous differential equation is

$$y'' - 2y' + 2y = 2,\ 0 < x < \pi \tag{1.21}$$

Let $y(x) = e^{mx}$ (m being a constant) be a trial solution of the corresponding homogenous differential equation of (1.21). Then its auxiliary equation is $m^2 - 2m + 2 = 0$
or, $m = 1 \pm i$. So the complementary function is

$$y_c(x) = e^x(c_1 \cos x + c_2 \sin x), \tag{1.22}$$

where c_1, c_2 are arbitrary constants.
And particular solution is

$$y_p(x) = \frac{2}{D^2 - 2D + 2} = 1 \tag{1.23}$$

So the general solution is

$$y(x) = y_c(x) + y_p(x) = e^x(c_1 \cos x + c_2 \sin x) + 1, \ 0 \leq x \leq \pi \tag{1.24}$$

where c_1, c_2 are arbitrary constants.

Case I: When $y(0) = 1, \ y(\pi) = 1$
The Case-I of the Example 1.4 is the associated homogeneous problem of (1.21) with these subsidiary conditions $y(0) = 1, \ y(\pi) = 1$. As the associated homogeneous problem has infinite solution so, the non-homogeneous boundary value problem (1.21) has same if the solution (1.24) satisfy the subsidiary conditions $y(0) = 1$ and $y(\pi) = 1$ by Theorem 1.5. Using the these subsidiary conditions in (1.24) we get, $c_1 = 0$ but no value of c_2. Hence, the solution is $y = c_2 e^x \sin x + 1, \ 0 \leq x \leq \pi$ and this solution is valid for any values of c_2. So in this case there are infinite solutions.

Case II: When $y(0) = 2, \ y(\frac{\pi}{2}) = 2$.
The Case-II of the Example 1.4 is the associated homogeneous problem of (1.21) with these subsidiary conditions $y(0) = 2, \ y(\frac{\pi}{2}) = 2$. As the associated homogeneous problem has unique solution so, the non-homogeneous boundary value problem (1.21) has also unique solution if the solution (1.24) satisfy subsidiary conditions $y(0) = 2, \ y(\frac{\pi}{2}) = 2$ by Theorem 1.5. Using these subsidiary conditions in (1.24) we get, $c_1 = 1, \ c_2 = e^{-\frac{\pi}{2}}$. So in this case there are unique solution $y(x) = e^x(\cos x + e^{-\frac{\pi}{2}} \sin x) + 1, \ 0 \leq x \leq \frac{\pi}{2}$.

1.3.2 Eigenvalue Problems

Let us consider the homogeneous boundary-value problem as started in section (1.3). Also from the theorem-1.4, we see that, this problem has non trivial solution if condition (1.13) is satisfied. From (1.11), we see that for certain values of λ as determined by (1.13) for which non trivial solution. These values of λ for which non-trivial solution to a homogeneous boundary value problems do exist are called **eigenvalues** and the corresponding non trivial solutions are known as **eigenfunctions**.

Example 1.6 Find the eigenvalues and eigenfunctions of
$\frac{d^2y}{dx^2} + \lambda y = 0, \ 0 < x < 1; \ y(0) = 0, \ y(1) = 0$ **V.U(H) : 2016, NET(MS): (June)2012**

Solution: The given homogeneous differential equation is

$$\frac{d^2y}{dx^2} + \lambda y = 0, \ 0 < x < 1 \tag{1.25}$$

Let $y(x) = e^{mx}$ (m being a constant) be a trial solution of (1.25). Then the A.E. of (1.25) is $m^2 + \lambda = 0$ or $m = \pm\sqrt{-\lambda}$. We now consider the three cases $\lambda = 0, \lambda < 0 \ \lambda > 0$ separately.

Case-I : When $\lambda = 0$

When $\lambda = 0$, then $m = 0, 0$. So the solution of the equation may be written as $y = c_1 + c_2 x$. Using boundary condition, we get $c_1 = c_2 = 0$. Therefore $y = 0$ is the only solution of (1.25) on [0, 1]. But, the solution $y = 0$ is a trivial solution. So, $\lambda = 0$ is not an eigenvalue of (1.25).

Case-II : When $\lambda < 0$

When $\lambda < 0$, $\lambda = -k^2$, so the auxiliary equation becomes $m^2 = k^2$ and hence $m = \pm k$ and the solution will be $y = c_1 e^{kx} + c_2 e^{-kx}$. Since $\begin{vmatrix} e^{kx} & e^{-kx} \\ ke^{kx} & -ke^{-kx} \end{vmatrix} = -2k \neq 0$. So $y_1 = e^{kx}$ and $y_2 = e^{-kx}$ are two independent solutions of the given equation. Then the boundary condition give $c_1 + c_2 = 0$ and $c_1 e^k + c_2 e^{-k} = 0$. Since $\begin{vmatrix} 1 & 1 \\ e^k & e^{-k} \end{vmatrix} = e^{-k} - e^k \neq 0$. So the said homogenous system of equations of c_1, c_2 has unique solutions $c_1 = c_2 = 0$. Therefore $y = 0$ is the only solution of (1.25) on [0, 1]. But, the solution $y = 0$ is again a trivial solution. So, $\lambda < 0$ is not an eigenvalue of (1.25).

Case-III : When $\lambda > 0$

In this case the auxiliary equation $m^2 + \lambda = 0$ given $m = \pm i\sqrt{\lambda}$ and the solution is $y = c_1 \cos \sqrt{\lambda}\, x + c_2 \sin \sqrt{\lambda}\, x$. Now using the boundary conditions we get $c_1 = 0$ and $c_2 \sin \sqrt{\lambda} = 0$. Now, if $c_2 = 0$, then the equation (1.25) has only trivial solution $y(x) = 0$ on $[0, 1]$. But to get non-trivial solution, let $c_2 \neq 0$ then $\sin \sqrt{\lambda} = 0$ which implies $\sqrt{\lambda} = n\pi$, $n = 0, \pm 1, \pm 2 \cdots$ or $\lambda = n^2\pi^2$, $n = 1, 2, 3, \cdots$ $[n \neq 0,$ since $\lambda \neq 0]$ which are the eigenvalues of this boundary value problem (1.25). The corresponding eigenfunctions are $y_n(x) = A_n \sin n\pi\, x$, $0 < x < 1$ where the arbitrary constants A_n are different for different values of $n = 1, 2, 3, \cdots$. Also, we see that the eigenvalue λ does not change the sign and it is always positive.

Note: Given that $y(0) = 0$, $y(1) = 0$, so $y(x)$ is trivial at $x = 0, 1$. Hence the eigenfunctions $y_n(x) = A_n \sin n\pi\, x$ are taken the values on the open interval $(0, 1)$ for $n = 1, 2, 3, \cdots$.

1.4 Orthogonal set of functions

Definition 1.1 (**Orthogonality of the two functions**) Two functions ϕ, ψ are said to be orthogonal functions over the interval [a, b] if

$$\int_a^b \phi(x)\psi(x)dx = 0$$

Definition 1.2 (**Orthogonal set of functions**) A set of functions $\{\phi_i\}$,$(i = 0, 1, 2, \cdots, n)$ is said to be orthogonal set of functions over the interval [a, b] if

$$\int_a^b \phi_i(x)\phi_j(x)dx = 0,\ i \neq j.$$

Definition 1.3 (**Orthogonality with respect to a weight function**) Two functions ϕ, ψ are said to be orthogonal functions over the interval [a, b] with respect to the weight function W if

$$\int_a^b W(x)\phi(x)\psi(x)dx = 0$$

Definition 1.4 (**Orthogonal set of functions with respect to a weight function**) A set of functions $\{\phi_i\}$,$(i = 0, 1, 2, \cdots, n)$ is said to be orthogonal set of functions over the interval [a, b] with respect to the weight function W if

$$\int_a^b W(x)\phi_i(x)\phi_j(x)dx = 0,\ i \neq j.$$

Example 1.7 Show that the set of functions $\{\cos nx,\ (n = 0, 1, 2, \cdots)\}$ is orthogonal on the interval $-\pi \leq x \leq \pi$.

Solution. Here the given functions are ϕ_n defined by $\phi_n(x) = \cos nx,\ n = 0, 1, 2, \cdots$. For $m \neq n$, we have

$$\int_{-\pi}^{\pi} \phi_m(x)\phi_n(x)dx = \int_{-\pi}^{\pi} \cos mx \cos nx dx$$

$$= \frac{1}{2}\int_0^{\pi} 2\cos mx \cos nx dx = \frac{1}{2}\Big[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\Big]_{-\pi}^{\pi} = 0.$$

Therefore, the given set of functions is orthogonal on the interval $-\pi \leq x \leq \pi$.

Example 1.8 Show that the functions $1 - x,\ 1 - 2x + \frac{x^2}{2}$ are orthogonal with respect to e^{-x} on the interval $0 \leq x < \infty$.

Solution. Here the given functions are $1 - x,\ 1 - 2x + \frac{x^2}{2}$ and the given weight function is e^{-x}. Let $\phi(x) = 1 - x,\ \psi(x) = 1 - 2x + \frac{x^2}{2}$ and $W(x) = e^{-x}$.

$$\text{Now,} \quad \int_0^{\infty} W(x)\phi(x)\psi(x)dx = \int_0^{\infty} e^{-x}(1-x)(1-2x+\frac{x^2}{2})dx = \int_0^{\infty} e^{-x}(1-3x+\frac{5x^2}{2}-\frac{x^3}{2})dx$$

$$= \int_0^{\infty} e^{-x}x^0 dx - 3\int_0^{\infty} e^{-x}x dx + \frac{5}{2}\int_0^{\infty} e^{-x}x^2 dx - \frac{1}{2}\int_0^{\infty} e^{-x}x^3 dx = 0.$$

Therefore $\int_0^{\infty} W(x)\phi(x)\psi(x)dx = 0$. Hence, the functions $1 - x,\ 1 - 2x + \frac{x^2}{2}$ are orthogonal with respect to e^{-x} on the interval $0 \leq x < \infty$.

1.5 Sturm-Liouville problems

A second order Sturm-Liouville problem is a homogeneous boundary value problem of the form

$$\frac{d}{dx}\{p(x)\frac{dy}{dx}\} + \{q(x) + \lambda\, r(x)\}y = 0 \tag{1.26}$$

with the boundary conditions

$$A_1 y(a) + B_1 y'(a) = 0, \tag{1.27}$$

$$A_2 y(b) + B_2 y'(b) = 0 \tag{1.28}$$

where p, q, r and p' are all real valued continuous functions on $[a, b]$ and both p and r are positive on $[a, b]$ and λ is a parameter independent of x. It is to be noted that $p,\ q,\ r$ are all independent of λ and also the constants $A_1,\ B_1,\ A_2,\ B_2$ are also independent of λ. Also it is to be noted that A_1 and A_2 are not both zero and B_1 and B_2 are also not both zero.
This type of boundary value problem is also called a Sturm-Liouville System or Regular Sturm-Liouville System.

Let us consider three supplementary conditions with periodic end conditions

$$p(a) = p(b) \tag{1.29}$$

$$y(a) = y(b) \tag{1.30}$$

$$y'(a) = y'(b). \tag{1.31}$$

Then the second order differential equation (1.26) with three supplementary conditions (1.29)-(1.31) is called a Periodic Sturm-Liouville problem.

Example 1.9 Express the boundary value problem $y'' + \lambda\, y = 0,\ 0 < x < \pi$ which satisfies the boundary conditions to $y(0) = 0$; $y'(\pi) = 0$ into a Sturm-Liouville problem.

Solution: The given homogeneous differential equation is

$$y'' + \lambda y = 0,\ 0 < x < \pi \text{ with } y(0) = 0,\ y'(\pi) = 0. \tag{1.32}$$

From the equation (1.32), we see that $p(x) = 1,\ q(x) = 0,$ and $r(x) = 1,$ and then the given equation can be put in the form

$$\frac{d}{dx}\Big(1.\frac{dy}{dx}\Big) + \Big(0 + \lambda.1\Big)y = 0 \text{ with } 1.y(0) + 0.y'(0) = 0,\ 0.y(\pi) + 1.y'(\pi) = 0. \tag{1.33}$$

where $p(x) = 1 > 0,$ and $r(x) = 1 > 0$ for all $x \in [0, \pi]$.
Thus the problem can be treated as Sturm-Liouville problem.

Example 1.10 Express the boundary value problem $y'' + \lambda\, y = 0,\ -\pi < x < \pi$ which satisfies the boundary conditions to $y(-\pi) = y(\pi)$; $y'(-\pi) = y'(\pi)$ into a periodic Sturm-Liouville problem.

Solution: The given homogeneous differential equation is

$$y'' + \lambda y = 0,\ -\pi < x < \pi \text{ with } y(-\pi) = y(\pi),\ y'(-\pi) = y'(\pi). \tag{1.34}$$

From the equation (1.34), we see that $p(x) = 1$, $q(x) = 0$, and $r(x) = 1$, and then the given equation can be put in the form

$$\frac{d}{dx}\Big(1.\frac{dy}{dx}\Big) + \Big(0 + \lambda.1\Big)y = 0 \text{ with } p(-\pi) = p(\pi); \qquad y(-\pi) = y(\pi), \qquad y'(-\pi) = y'(\pi).$$

where $p(x) = 1 > 0$, and $r(x) = 1 > 0$ for all $x \in [-\pi, \pi]$.
Thus the problem can be treated as periodic Sturm-Liouville problem.

1.6 Conversion of a Second Order Linear Differential Equation to Sturm-Liouville Form

Let us consider a second order linear differential equation of the form

$$p_0(x)\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_2(x)y + \lambda\, R(x)y = 0 \tag{1.35}$$

on $[a, b]$, where $p_0(x)(\neq 0)$ and $R(x)$ are positive in the interval where the problem is considered. Multiplying the equation (1.35) by

$$I(x) = e^{\int \frac{p_1(x)}{p_0(x)}dx} \tag{1.36}$$

we get, $I(x)P_0(x)\frac{d^2y}{dx^2} + I(x)P_1(x)\frac{dy}{dx} + I(x)P_2(x)y + \lambda\, I(x)R(x)y = 0$

$\Rightarrow P_0(x)[I(x)\frac{d^2y}{dx^2} + I(x)\frac{P_1(x)}{P_0(x)}\frac{dy}{dx}] + I(x)p_2(x)y + \lambda\, I(x)R(x)y = 0$

$\Rightarrow P_0(x)\frac{d}{dx}[I(x)\frac{dy}{dx}] + I(x)P_2(x)y + \lambda\, I(x)R(x)y = 0,$ [since $\frac{dI(x)}{dx} = e^{\int \frac{p_1(x)}{p_0(x)}dx}.\frac{P_1(x)}{P_0(x)} = I(x)\frac{P_1(x)}{P_0(x)}$].

Now dividing by $P_0(x)$, we get $\frac{d}{dx}[I(x)\frac{dy}{dx}] + [I(x)\frac{P_2(x)}{P_0(x)} + \lambda\, \frac{I(x)R(x)}{P_0(x)}]y = 0.$

Now putting $p(x) = I(x)$, $q(x) = I(x)\frac{P_2(x)}{P_0(x)}$ and $r(x) = \frac{I(x)R(x)}{P_0(x)}$, we get,

$\frac{d}{dx}\{P(x)\frac{dy}{dx}\} + \{q(x) + \lambda\, r(x)\}y = 0,$ on $[a, b]$, which is of Sturm-Liouville form.

1.7 Some properties of regular Sturm-Liouville problems

Property 1.1 The eigenvalue of Sturm-Liouville problems are all real.

Property 1.2 There exists an infinite number of characteristic values(eigenvalues) λ_n of the given problem. These characteristic values λ_n can be arranged in a monotonic increasing sequence $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$ such that $\lambda_n \to \infty$ $n \to \infty$.

Property 1.3 For each eigenvalues of Sturm-Liouville problems (1.26), there exits one and only one linearly independent eigenfunction.

Property 1.4 The eigenfunctions corresponding to different eigenvalues are orthogonal with respect to weight function **r**.

Theorem 1.6 Let the coefficients $p(x), q(x)$ and $r(x)$ in the Sturm-Liouville system be continuous in $[a, b]$. Let the eigen functions ϕ_j and ϕ_k corresponding to λ_j and λ_k be continuously differentiable. Then ϕ_j and ϕ_k are orthogonal *w.r.t.* the weight function **r** in $[a, b]$.

Proof. We have Sturm-Liouville system

$$\frac{d}{dx}\{p(x)\frac{dy}{dx}\} + \{q(x) + \lambda\, r(x)\}y = 0 \tag{1.37}$$

Since ϕ_j and ϕ_k are the eigen functions of (1.37) corresponding to the eigen values λ_j and λ_k respectively, thus we have

$$\frac{d}{dx}\{p(x)\frac{d\phi_j}{dx}\} + \{q(x) + \lambda_j\, r(x)\}\phi_j = 0 \tag{1.38}$$

and
$$\frac{d}{dx}\{p(x)\frac{d\phi_k}{dx}\} + \{q(x) + \lambda_k\, r(x)\}\phi_k = 0 \tag{1.39}$$

Multiplying to the equation (1.38) by ϕ_k and to the equation (1.39) by ϕ_j and then substituting we get,

$$\begin{aligned}
& \phi_k\frac{d}{dx}\{p\frac{d\phi_j}{dx}\} + \{q + \lambda_j\, r\}\phi_j\phi_k - \phi_j\frac{d}{dx}\{p\frac{d\phi_k}{dx}\} - \{q + \lambda_k\, r\}\phi_j\phi_k = 0 \\
\Rightarrow\quad & r(\lambda_j - \lambda_k)\phi_j\phi_k = \phi_j\frac{d}{dx}\{p\frac{d\phi_k}{dx}\} - \phi_k\frac{d}{dx}\{p\frac{d\phi_j}{dx}\} \\
\Rightarrow\quad & r(\lambda_j - \lambda_k)\phi_j\phi_k = \frac{d}{dx}\Big[(p\phi_j\frac{d\phi_k}{dx}) - (p\frac{d\phi_j}{dx}\phi_k)\Big]
\end{aligned}$$

Now integrating to the above *w.r.t.* x within the limits a to b, we get

$$\begin{aligned}
(\lambda_j - \lambda_k)\int_a^b \phi_j\phi_k r(x)dx &= \Big[(p\phi_j\frac{d\phi_k}{dx}) - (p\frac{d\phi_j}{dx}\phi_k)\Big]_a^b \\
&= p(b)(\phi_j(b)\phi_k'(b) - \phi_j'(b)\phi_k(b)) - p(a)(\phi_j(a)\phi_k'(a) - \phi_j'(a)\phi_k(a))
\end{aligned} \tag{1.40}$$

Now the supplementary conditions of Sturm-Liouville system are

$$A_1\phi_j(a) + B_1\phi_j'(a) = 0, \tag{1.41}$$
$$A_2\phi_j(b) + B_2\phi_j'(b) = 0 \tag{1.42}$$

and
$$A_1\phi_k(a) + B_1\phi_k'(a) = 0, \tag{1.43}$$
$$A_2\phi_k(b) + B_2\phi_k'(b) = 0 \tag{1.44}$$

Multiplying to the equation (1.42) by $\phi_k(b)$ and to (1.44) by $\phi_j(b)$ and then substituting we get

$$B_2\Big[\phi_j'(b)\phi_k(b) - \phi_k'(b)\phi_j(b)\Big] = 0$$
$$\Rightarrow\quad \Big[\phi_j'(b)\phi_k(b) - \phi_k'(b)\phi_j(b)\Big] = 0, \qquad (\because B_2 \neq 0) \tag{1.45}$$

Similarly multiplying to the equation (1.41) by $\phi_k(a)$ and to (1.43) by $\phi_j(a)$ and then substituting we get

$$A_2\Big[\phi_j'(a)\phi_k(a) - \phi_k'(a)\phi_j(a)\Big] = 0$$
$$\Rightarrow\quad \Big[\phi_j'(a)\phi_k(a) - \phi_k'(a)\phi_j(a)\Big] = 0, \qquad (\because A_2 \neq 0) \tag{1.46}$$

Using (1.45) and (1.46) in (1.40), we have

$$(\lambda_j - \lambda_k)\int_a^b \phi_j\phi_k r(x)dx = 0 \tag{1.47}$$

Since λ_j and λ_k are distinct eigen values, so $\lambda_j \neq \lambda_k$, therefore from (1.47), we get

$$\int_a^b \phi_j(x)\phi_k(x)r(x)dx = 0 \tag{1.48}$$

which shows that ϕ_j and ϕ_k are orthogonal *w.r.t.* the weight function **r** on $[a, b]$. Hence the theorem.

Theorem 1.7 All the eigen values of a regular Sturm-Liouville system with $\mathbf{r(x)} > 0$, are real.

Proof. Let ϕ_j and ϕ_k be the eigen functions of the regular Sturm-Liouville system (1.37) corresponding to the eigen values λ_j and λ_k respectively. Then using similar way of the Theorem 1.6, we have (like equation (1.47))

$$(\lambda_j - \lambda_k)\int_a^b \phi_j\phi_k r(x)dx = 0 \tag{1.49}$$

Let us assume that $\lambda_j = \alpha + i\beta$ corresponding to $\phi_j = u + iv$. Then as the coefficients of Sturm-Liouville equation are real, the complex conjugate of λ_j is also an eigen value. Thus there exists an eigen function $\phi_k = u - iv = \bar{\phi}_j$ corresponding to the eigen value $\lambda_k = \alpha - i\beta = \bar{\lambda}_j$. Using the above conditions, in equation (1.49), we get

$$\Big[(\alpha + i\beta) - (\alpha - i\beta)\Big]\int_a^b (u + iv)(u - iv)r(x)dx = 0$$

$$\Rightarrow \quad 2i\beta\int_a^b (u^2 + v^2)r(x)dx = 0$$

Since $r(x)$ is positive and $u^2 + v^2$ is positive. Therefore β must be equal to zero. Hence eigen values of regular Sturm-Liouville system are real.

Theorem 1.8 The eigen functions of the periodic Sturm-Liouville system in $[a, b]$ are orthogonal *w.r.t.* the weight function **r** in $[a, b]$.

Proof. We have the periodic Sturm-Liouville system as

$$\frac{d}{dx}\{p(x)\frac{dy}{dx}\} + \{q(x) + \lambda\, r(x)\}y = 0 \tag{1.50}$$

with three supplementary conditions (1.29) - (1.31). Let ϕ_j and ϕ_k be the eigen functions of the periodic Sturm-Liouville system (1.50) corresponding to the eigen values λ_j and λ_k respectively.

Then using similar way of the Theorem 1.6, we have (like equation (1.40))

$$(\lambda_j - \lambda_k)\int_a^b \phi_j\phi_k r(x)dx = p(b)(\phi_j(b)\phi_k'(b) - \phi_j'(b)\phi_k(b)) - p(a)(\phi_j(a)\phi_k'(a) - \phi_j'(a)\phi_k(a)) \quad (1.51)$$

Then using the three supplementary conditions (1.29) - (1.31)of periodic Sturm-Liouville system, we have

$$p(a) = p(b) \quad (1.52)$$

$$\text{and} \quad \phi_j(a) = \phi_j(b), \quad \phi_j'(a) = \phi_j'(b) \quad (1.53)$$

$$\text{and} \quad \phi_k(a) = \phi_k(b), \quad \phi_k'(a) = \phi_k'(b) \quad (1.54)$$

Using (1.52), (1.53) and (1.54) in (1.51), we get

$$(\lambda_j - \lambda_k)\int_a^b \phi_j\phi_k r(x)dx = 0 \quad (1.55)$$

Since λ_j and λ_k are distinct eigenvalues, so $\lambda_j \neq \lambda_k$, therefore from (1.55), we get

$$\int_a^b \phi_j(x)\phi_k(x)r(x)dx = 0 \quad (1.56)$$

which shows that ϕ_j and ϕ_k are orthogonal *w.r.t.* the weight function **r** on $[a, b]$. Hence the theorem.

Theorem 1.9 If $\phi_1(x)$ and $\phi_2(x)$ are any two solutions of the Sturm-Liouville equation on $[a, b]$, then $p(x)W[x; \phi_1, \phi_2]$ = constant, where W is the Wronskian and $W[x; \phi_1, \phi_2] = |\phi_1\phi_2' - \phi_1'\phi_2|$.

Proof. Since ϕ_1 and ϕ_2 are any two solutions of the Sturm-Liouville equation, then

$$\frac{d}{dx}\{p(x)\frac{d\phi_1}{dx}\} + \{q(x) + \lambda\, r(x)\}\phi_1 = 0 \quad (1.57)$$

$$\text{and} \quad \frac{d}{dx}\{p(x)\frac{d\phi_2}{dx}\} + \{q(x) + \lambda\, r(x)\}\phi_2 = 0 \quad (1.58)$$

Multiplying to the equation (1.57) by ϕ_2 and to (1.58) by ϕ_1 and then substituting we get

$$\phi_1\frac{d}{dx}\{p(x)\frac{d\phi_2}{dx}\} - \phi_2\frac{d}{dx}\{p(x)\frac{d\phi_1}{dx}\} = 0$$

$$\Rightarrow \quad \frac{d}{dx}\Big[(p\phi_2')\phi_1 - (p\phi_1')\phi_2\Big] = 0$$

Integrating above *w.r.t.* x to the limits a to x, we get,

$$\int_a^x d\Big[(p\phi_2')\phi_1 - (p\phi_1')\phi_2\Big] = 0$$

$$\Rightarrow \quad \Big[(p\phi_2')\phi_1 - (p\phi_1')\phi_2\Big]_a^x = 0$$

$$\Rightarrow \quad p(x)\Big[\phi_2'(x)\phi_1(x) - \phi_1'(x)\phi_2(x)\Big] - p(a)\Big[\phi_2'(a)\phi_1(a) - \phi_1'(a)\phi_2(a)\Big] = 0$$

$$\Rightarrow \quad p(x)W[x; \phi_1, \phi_2] = \text{constant.}$$

1.8 Worked Out Examples

Example 1.11 Solve: $\frac{d^2y}{dx^2} - 9y = 0$; $y(0) = 0$ and $y'(0) = 1$.

Solution: The given homogeneous differential equation is

$$\frac{d^2y}{dx^2} - 9y = 0 \tag{1.59}$$

Let $y(x) = e^{mx}$ (m being a constant) be a trial solution of (1.59). Then the A.E. of (1.59) is $m^2 - 9 = 0$ or $m = \pm 3$. So the general solution is $y = c_1e^{3x} + c_2e^{-3x}$ and $y' = 3c_1e^{3x} - 3c_2e^{-3x}$. Then using conditions, we get, $c_1 + c_2 = 0$ and $3c_1 - 3c_2 = 1$. So, $c_1 = \frac{1}{6}$ and $c_2 = -\frac{1}{6}$. Hence the solution is $y = \frac{1}{6}(e^{3x} - e^{-3x})$.

Example 1.12 Express the boundary value problem $x^2y'' + xy' + \lambda y = 0$, $1 < x < e^{2\pi}; \lambda > 0$, which satisfies the boundary conditions to $y'(1) = 0$; $y'(e^{2\pi}) = 0$ into a Sturm-Liouville problem.

Solution: The given homogeneous differential equation is

$$x^2y'' + xy' + \lambda y = 0,\ 1 < x < e^{2\pi};\ \text{with}\ y'(1) = 0,\ \ y'(e^{2\pi}) = 0. \tag{1.60}$$

The given equation (1.60) can be put in the form

$$\frac{d}{dx}\Big(x.\frac{dy}{dx}\Big) + \Big(0 + \frac{\lambda}{x}\Big)y = 0\ \text{with}\ 0.y(1) + 1.y'(1) = 0,\ 0.y(e^{2\pi}) + 1.y'(e^{2\pi}) = 0. \tag{1.61}$$

where $p(x) = x > 0$, $q(x) = 0$ and $r(x) = \frac{1}{x} > 0$ for all $x \in [1,\ e^{2\pi}]$.
Thus the problem can be treated as Sturm-Liouville problem.

Example 1.13 Find the eigenvalues and the eigenfunctions for the differential equation $y'' + \lambda y = 0\,, 0 < x < \pi$, subject to $y(0) = 0$; $y(\pi) = 0$ and also show that the set of eigenfunctions are orthogonal on the interval $0 < x < \pi$. **NET(Dec.)(MS)-2017, ,V.H.-2015**

Solution: The given homogeneous differential equation is

$$y'' + \lambda y = 0,\ 0 < x < \pi \tag{1.62}$$

Let $y(x) = e^{mx}$ (m being a constant) be a trial solution of (1.62). Then the A.E. of (1.62) is $m^2 + \lambda = 0$ or $m = \pm\sqrt{-\lambda}$.

Case I: If $\lambda = 0$, the general solution of the equation is $y = c_1 + c_2x$, c_1, c_2 being arbitrary constants. But by using the boundary conditions $y(0) = 0$ and $y(\pi) = 0$, we get, $c_1 = c_2 = 0$ and hence $y = 0$ is the solution on $[0, \pi]$ of (1.62). But, the said solution is a trivial solution in the closed interval $[0, \pi]$. Thus $\lambda = 0$ is not an eigenvalue of (1.62).

Case II: If $\lambda < 0$, the solution is $y = c_1e^{\sqrt{-\lambda}x} + c_2e^{-\sqrt{-\lambda}x}$ where $-\lambda$ is positive. Then the boundary condition $y(0) = 0$ and $y(\pi) = 0$, give $c_1 + c_2 = 0$ and $c_1e^{\sqrt{-\lambda}\pi} + c_2e^{-\sqrt{-\lambda}\pi} = 0$ i.e, $c_1 = c_2 = 0$ and consequently the equation (1.62) has a trivial solution $y = 0$. Thus $\lambda < 0$ are not eigenvalues of (1.62).

Case III: If $\lambda > 0$, the solution is $y = c_1 \cos \sqrt{\lambda}x + c_2 \sin \sqrt{\lambda}x$. The condition $y(0) = 0$ gives $c_1 = 0$ and the condition $y(\pi) = 0$ gives $c_2 \sin(\sqrt{\lambda}\pi) = 0$. Then to get non-trivial solution let $c_2 \neq 0$ and then, $\sin(\sqrt{\lambda}\pi) = 0$ which implies $\sqrt{\lambda}\pi = n\pi$ or, $\sqrt{\lambda} = n$. So the eigenvalues of (1.62) are $\lambda = n^2$ for $n = 1, 2, 3, \cdots$ and corresponding eigenfunctions are $\phi_n(x) = A_n \sin nx\,, 0 < x < \pi$, where $n = 1, 2, 3, \cdots$.

Next, the eigenfunctions are $\phi_n(x) = A_n \sin nx$, $0 < x < \pi$, $n = 1, 2, 3, \cdots$. For $m \neq n$ and $n, m = 1, 2, 3, \cdots$, we have

$$\int_0^\pi \phi_m(x)\phi_n(x)dx = \int_0^\pi A_m A_n \sin mx \sin nx dx$$

$$= \frac{1}{2}\int_0^\pi A_n A_m 2 \sin mx \sin nx dx = \frac{A_n A_m}{2}\Big[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\Big]_0^\pi = 0.$$

Therefore, the given set of eigenfunctions is orthogonal on the interval $0 < x < \pi$.

Example 1.14 Find the set of all eigenvalues and corresponding eigenfunctions of

$$y'' + \lambda y = 0,\ 0 < x < \frac{\pi}{2},\ \text{with } y'(0) = 0,\ y'(\frac{\pi}{2}) = 0$$

GATE(MA)-04

Solution: The given homogeneous differential equation is

$$y'' + \lambda y = 0,\ 0 < x < \frac{\pi}{2} \tag{1.63}$$

Let $y(x) = e^{mx}$ (m being a constant) be a trial solution of (1.63). Then the A.E. of (1.63) is $m^2 + \lambda = 0$ or $m = \pm\sqrt{-\lambda}$.

Case I: If $\lambda = 0$, the general solution of the equation is $y = c_1 + c_2 x$, c_1, c_2 being arbitrary constants. But by using the boundary conditions $y'(0) = 0$ and $y'(\frac{\pi}{2}) = 0$, we get, $c_2 = 0$ and hence $y(x) = c_1$ is the solution on $[0, \frac{\pi}{2}]$ of (1.63). Thus $\lambda = 0$ is an eigenvalue of (1.63) and corresponding eigenfunctions is $y(x) = c_1$ where c_1 is arbitrary.

Case II: If $\lambda < 0$, the solution is $y(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$ where $-\lambda$ is positive. Then the boundary condition $y'(0) = 0$ and $y'(\frac{\pi}{2}) = 0$, give $c_1 = c_2 = 0$ and consequently the equation (1.63) has a trivial solution $y = 0$. Thus $\lambda < 0$ are not eigenvalues of (1.63).

Case III: If $\lambda > 0$, the solution is $y(x) = c_1 \cos \sqrt{\lambda}x + c_2 \sin \sqrt{\lambda}x$. The condition $y'(0) = 0$ gives $c_2 = 0$ and the condition $y'(\frac{\pi}{2}) = 0$ gives $c_1 \sqrt{\lambda} \sin(\sqrt{\lambda} \cdot \frac{\pi}{2}) = 0$. Then to get non-trivial solution let $c_1 \neq 0$ and then, $\sin(\frac{\sqrt{\lambda}\pi}{2}) = 0$ which implies $\frac{\sqrt{\lambda}\pi}{2} = n\pi$, $n = 0, 1, 2, 3, \cdots$ or, $\sqrt{\lambda} = 2n$. So the eigenvalues of (1.63) are $\lambda = 4n^2$ for $n = 0, 1, 2, 3, \cdots$ and corresponding eigenfunctions are $\phi_n(x) = A_n \cos nx\,, 0 \leq x < \frac{\pi}{2}$, where $n = 0, 1, 2, 3, \cdots$.

Example 1.15 Find the eigenvalues and the eigenfunctions for the differential equation $y'' + \lambda\, y = 0$, $0 < x < \pi$; $\lambda > 0$, which satisfies the boundary conditions to $y(0) = 0$; $y'(\pi) = 0$. Find also the difference between the least two eigenvalues of the said boundary value problem.
NET(MS): (June)2013, **GATE(MA):2016**

Solution: The given homogeneous differential equation is

$$y'' + \lambda y = 0,\ 0 < x < \pi \tag{1.64}$$

Let $y(x) = e^{mx}$ (m being a constant) be a trial solution of (1.64). Then the A.E. of (1.64) is $m^2+\lambda = 0$ or $m = \pm\sqrt{-\lambda}$. It can be shown easily as previous examples that λ is not an eigenvalue when $\lambda \le 0$. When $\lambda > 0$ the solution of the above problem is $y = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$, where c_1, c_2 are arbitrary constants. Applying the boundary condition $y(0) = 0$, we get $c_1 = 0$. Also the condition $y'(\pi) = 0$ gives $c_2\lambda\cos(\sqrt{\lambda}\pi) = 0$ which implies $c_2\cos(\sqrt{\lambda}\pi) = 0$ as $\lambda > 0$. Then $c_2 = 0$ gives $y(x) = 0$ which is a trivial solution of (1.64). So, to get non-trivial solution of (1.64), we must have $c_2 \neq 0$ and consequently $\cos\sqrt{\lambda}\pi = 0$ or $\sqrt{\lambda} = \frac{2n-1}{2}, n = 1, 2, 3\cdots$ and hence the eigenvalues (1.64) are $\lambda_n = \frac{(2n-1)^2}{4}, n = 1, 2, 3\cdots$ and the corresponding eigenfunctions are $\phi_n(x) = A_n \sin\frac{2n-1}{2}x$, $n = 1, 2, \cdots, 0 < x < \pi$.
The difference between the least two eigenvalues is $\frac{9}{4} - \frac{1}{4} = 2$.

Example 1.16 Find the eigenvalues and the eigenfunctions for the differential equation $\frac{d}{dx}(x\frac{dy}{dx}) + \frac{\lambda y}{x} = 0$, $1 < x < e^{\pi}$ $(\lambda > 0)$, which satisfies the boundary conditions $y(1) = 0$; $y(e^{\pi}) = 0$.

Solution: The equation can be written as

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{\lambda}{x}y = 0 \Rightarrow x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \lambda y = 0.$$

This being homogeneous linear equation, (Cauchy-Euler form), we put $z = \log x$ or $x = e^z$
Then the equation transforms to

$$\{D(D-1) + D + \lambda\}y = 0 \Rightarrow (D^2 + \lambda)y = 0 \text{ when } D \equiv \frac{d}{dz} \tag{1.65}$$

The given boundary conditions $y(x) = 0$ at $x = 1$ and $y(x) = 0$ at $x = e^{\pi}$ become $y(z) = 0$ at $z = \log 1 = 0$ and $y(z) = 0$ at $z = \log(e^{\pi}) = \pi$ respectively. Hence the problem becomes to find the solution of $\frac{d^2y}{dz^2} + \lambda y = 0$, $0 < z < \pi$, $y(0) = 0$, $y(\pi) = 0$. Now, proceeding as previous Example 1.13, the eigenvalues are $\lambda_n = n^2$ and the corresponding eigenfunction are

$$\phi_n(n) = A_n \sin nz = A_n \sin(n \log x), \text{ when in both cases } n = 1, 2, 3, \cdots, 0 < x < e^{\pi}.$$

Example 1.17 Find the eigenvalues and the eigenfunctions of the Sturm-Liouville problem $y'' + \lambda y = 0$, $0 < x < 1$, subject to $y(0) + y'(0) = 0$ and $y(1) + y'(1) = 0$. [C.U. (Hons.)-2002]

Solution: The given homogeneous differential equation is

$$y'' + \lambda y = 0,\ 0 < x < 1 \text{with } y(0) + y'(0) = 0 \text{ and } y(1) + y'(1) = 0. \tag{1.66}$$

Let $y(x) = e^{mx}$ (m being a constant) be a trial solution of (1.66). Then the A.E. of (1.66) is $m^2 + \lambda = 0$ or $m = \pm\sqrt{-\lambda}$.

Case I: If $\lambda = 0$, then the general solution of the equation is $y = c_1 + c_2x$, where c_1, c_2 being arbitrary constants. But by the boundary conditions $y(0) + y'(0) = 0$ and $y(1) + y'(1) = 0$, we get, $c_1 + c_2 = 0$ and $c_1 + 2c_2 = 0$ or $c_1 = c_2 = 0$ and hence the solution of (1.66) is $y = 0$ which is a trivial solution in the closed interval [0, 1]. Thus $\lambda = 0$ is not an eigenvalue of the problem (1.66).

Case II: If $\lambda < 0$, let $\lambda = -\mu^2$ where $\mu \neq 0$. Then the solution of (1.66)

$$y(x) = c_1 e^{\mu x} + c_2 e^{-\mu x} \tag{1.67}$$

$$\text{now } y'(x) = \mu c_1 e^{\mu x} - \mu c_2 e^{-\mu x} \tag{1.68}$$

Using the boundary conditions $y(0) + y'(0) = 0$ and $y(1) + y'(1) = 0$, from (1.67) and (1.68), we get,

$$c_1 + c_2 + \mu(c_1 - c_2) = 0 \text{ i.e., } c_1(1+\mu) + c_2(1-\mu) = 0 \tag{1.69}$$

$$\text{and} \quad c_1 e^{\mu} + c_2 e^{-\mu} + \mu(c_1 e^{\mu} - c_2 e^{-\mu}) = 0 \text{ i.e., } c_1 e^{\mu}(1+\mu) + c_2 e^{-\mu}(1-\mu)) = 0. \tag{1.70}$$

For non-zero values of c_1, c_2, we must have, $\begin{vmatrix} 1+\mu & 1-\mu \\ e^{\mu}(1+\mu) & e^{-\mu}(1-\mu) \end{vmatrix} = 0 \Rightarrow (1+\mu)(1-\mu)(e^{\mu} - e^{-\mu}) = 0 \Rightarrow \mu = \pm 1$. When $\mu = -1$, the equations (1.69) and (1.70) gives $c_2 = 0$ while c_1 will be arbitrary. So the equation (1.67) reduces to $y(x) = c_1 e^{-x}$ which is an eigenfunction and the corresponding eigenvalue is given by $\lambda = -\mu^2 = -(-1)^2 = -1$. When $\mu = 1$, the equations (1.69) and (1.70) gives $c_1 = 0$ while c_2 will be arbitrary. So the equation (1.67) reduces to $y(x) = c_2 e^{-x}$ which is an eigenfunction and the corresponding eigenvalue is given by $\lambda = -\mu^2 = -(-1)^2 = -1$. Taking $c_1 = c_2 = c$, $y(x) = ce^{-x}$ is an eigenfunction on $(0, 1)$ of (1.66) and $\lambda = -1$ is the corresponding eigenvalue, c being an arbitrary constant.

Case III: If $\lambda > 0$, let $\lambda = \mu^2$ where $\mu \neq 0$. Then the solution of (1.66)

$$y(x) = c_1 \cos(\mu x) + c_2 \sin(\mu x) \tag{1.71}$$

$$\text{and } y'(x) = -c_1 \sin(\mu x) + c_2 \cos(\mu x) \tag{1.72}$$

Using the boundary conditions $y(0) + y'(0) = 0$ and $y(1) + y'(1) = 0$, from (1.71) and (1.72), we get,

$$c_1 + c_2\mu = 0 \tag{1.73}$$

$$\text{and} \quad c_1 \cos\mu + c_2 \sin\mu - c_1\mu \sin\mu + c_2\mu\cos\mu = 0. \tag{1.74}$$

From (1.73), we get $c_1 = -c_2\mu$. With this value of c_1, (1.74) gives

$$-c_2\mu\cos\mu + c_2\sin\mu + c_2\mu^2\sin\mu + c_2\mu\cos\mu = 0$$

$$\Rightarrow \quad c_2(1+\mu^2)\sin\mu = 0$$

$$\Rightarrow \quad c_2 \sin\mu = 0 \left[\text{since} 1 + \mu^2 \neq 0\right].$$

If $c_2 = 0$, then (1.73) gives $c_1 = 0$. Hence (1.71) reduces to $y(x) = 0$ which is a trivial solution of (1.66). So to get non-trivial solution, let $c_2 \neq 0$. Then we have $\sin\mu = 0 \Rightarrow \mu = n\pi$, $n = 1, 2, 3\cdots$. Then (1.73) gives $c_1 = -c_2\mu = -c_2 n\pi$. Thus (1.71) reduces to $y(x) = c_2\{\sin(n\pi x) + \cos(n\pi x)\}$, $n = 1, 2, 3\cdots$ and then eigenvalues of (1.66) are $\lambda = \mu^2 = n^2\pi^2$, $n = 1, 2, 3, \cdots$. Hence, the required eigenfunctions y_n with the corresponding eigenvalues λ_n (1.66) are given by $y_n(x) = A_n\Big\{\sin(n\pi x) - n\pi\cos(n\pi x)\Big\}$, $0 \leq x \leq 1$, and $\lambda_n = n^2\pi^2$, $n = 1, 2, 3, \cdots$.

Example 1.18 If $\frac{d^2y}{dx^2} + \lambda y = 0$, $a < x < b$, with $y(a) = 0$ and $y(b) = 0$, then prove that eigenvalues are $\lambda = \frac{n^2\pi^2}{(b-a)^2}$, $n = 1, 2, 3, \cdots$ and corresponding eigenfunctions are $y_n(x) = A_n \sin\left(\frac{n\pi(x-a)}{b-a}\right)$, $a < x < b$, $n = 1, 2, 3, \cdots$.

Solution: The given homogeneous differential equation is

$$y'' + \lambda y = 0,\ a < x < b \tag{1.75}$$

Let $y(x) = e^{mx}$ (m being a constant) be a trial solution of (1.75). Then the A.E. of (1.75) is $m^2 + \lambda = 0$ or $m = \pm\sqrt{-\lambda}$.

Case I: If $\lambda = 0$, the general solution of the equation is $y = c_1 + c_2 x$, c_1, c_2 being arbitrary constants. But by using the boundary conditions $y(a) = 0$ and $y(b) = 0$, we get, $c_1 = c_2 = 0$ and hence $y(x) = 0$ is the trivial solution on $[a, b]$ of (1.75). Thus $\lambda = 0$ is not an eigenvalue of (1.75) on $[a, b]$.

Case II: If $\lambda < 0$, the solution is $y(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$ where $-\lambda$ is positive. Then the boundary condition $y(a) = 0$ and $y(b) = 0$, give $c_1 = c_2 = 0$ and consequently the equation (1.75) has a trivial solution $y = 0$. Thus $\lambda < 0$ are not eigenvalues of (1.75).

Case III: If $\lambda > 0$, let $\lambda = m^2$, then the solution of (1.75) is $y(x) = c_1 \cos mx + c_2 \sin mx,\ a < x < b$. Using $y(a) = 0,\ y(b) = 0$, we get, $c_1 \cos ma + c_1 \sin ma = 0,\ c_1 \cos mb + c_1 \sin mb = 0$. To get at least one non zero solution of c_1, c_2 for this system of homogenous equations, we get,

$$\begin{vmatrix} \cos ma & \sin ma \\ \cos mb & \sin mb \end{vmatrix} = 0 \Rightarrow \sin m(b-a) = 0 \Rightarrow m = \frac{n\pi}{b-a},\ n = 1, 2, 3, \cdots.$$

So eigenvalues are $\lambda = \frac{n^2\pi^2}{(b-a)^2},\ n = 1, 2, 3, \cdots$ and corresponding eigenfunctions are $y_n(x) = A_n \sin\left(\frac{n\pi(x-a)}{b-a}\right),\ a < x < b,\ n = 1, 2, 3, \cdots$.

Example 1.19 The boundary value problem, $\frac{d^2\phi}{dx^2} + \lambda\phi = x,\ 0 < x < 1;\ \phi(0) = 0,\ \frac{d\phi}{dx}(1) = 0$ is converted into $\phi(x) = g(x) + \lambda\int_0^1 k(x,\xi)\phi(\xi)d\xi$, where the kernel $k(x,\xi) = \begin{cases} \xi, & 0 < \xi < x \\ x, & x < \xi < 1 \end{cases}$, then find the value of $g(\frac{2}{3})$. **GATE(MA)-14**

Solution: Given that, $k(x,\xi) = \begin{cases} \xi, & 0 < \xi < x \\ x, & x < \xi < 1 \end{cases}$. So,

$$\begin{aligned} \phi(x) &= g(x) + \lambda\int_0^x k(x,\xi)\phi(\xi)d\xi + \lambda\int_x^1 k(x,\xi)\phi(\xi)\, d\xi \\ &= g(x) + \lambda\int_0^x \xi\phi(\xi)d\xi + \lambda\int_x^1 x\phi(\xi)d\xi. \end{aligned} \tag{1.76}$$

When $\phi(0) = 0 \Rightarrow g(0) + 0 + \lambda\int_0^1 0.\ \phi(\xi)\, d\xi = 0 \Rightarrow g(0) = 0$. Now, differentiating the equation (1.76), we get

$$\phi'(x) = g'(x) + \lambda\int_x^1 \phi(\xi)\, d\xi \tag{1.77}$$

Putting, $\phi'(1) = 0$, we get, $g'(1) = 0$. Again, differentiating the equation (1.77), we get

$$\begin{aligned} &\phi''(x) = g''(x) - \lambda\phi(x) \Rightarrow \phi''(x) + \lambda\phi(x) = g''(x) \\ &\Rightarrow g''(x) = x \Rightarrow g(x) = \frac{x^3}{6} + Cx + D,\ 0 \le x \le 1 \end{aligned} \tag{1.78}$$

where C and D are integrating constants. Putting $g(0) = 0$ and $g'(1) = 0$ in the equation(1.78) we get $D = 0, C = -\frac{1}{2}$. Therefore, $g(x) = \frac{x^3}{6} - \frac{x}{2},\ 0 \le x \le 1$ and also $g(\frac{2}{3}) = -0.28$.

1.9 Multiple Choice Questions

1. The set of all eigenvalues of the S-L problem $y'' + \lambda y = 0$ with $y'(0) = 0,\ y'(\frac{\pi}{2}) = 0$ is given by
(a)$\lambda = 2n,\ n = 1, 2, 3, \cdots$ (b)$\lambda = 2n, n = 0, 1, 2, \cdots$
(c)$\lambda = 4n^2, n = 1, 2, 3, \cdots$ (d)$\lambda = 4n^2, n = 0, 1, 2, \cdots$ **Gate(MA): 2004**
Ans. (d) is correct.
Hint. The solution of the differential equation $y'' + \lambda y = 0$ is $y(x) = a_1 + a_2x, \lambda = 0$, $y(x) = b_1e^{\sqrt{-\lambda}} + b_2e^{-\sqrt{-\lambda}}$, $\lambda < 0$ and $y(x) = c_1 \cos \sqrt{\lambda}x + c_2 \sin \sqrt{\lambda}x,\ \lambda > 0$. Using given boundary conditions, we get, a_1 is arbitrary for $\lambda = 0$, $b_1 = b_2 = 0$ for $\lambda < 0$ and $c_2 = 0,\ c_1 \neq 0 \Rightarrow \sin \sqrt{\lambda}\frac{\pi}{2} = 0 \Rightarrow \sqrt{\lambda}\frac{\pi}{2} = n\pi$ for $n = 1, 2, 3, \cdots$ or $\lambda = 4n^2$ for $n = 1, 2, 3, \cdots$ for $\lambda > 0$. Hence eigenvalues of the S-L problem is $\lambda = 4n^2, n = 0, 1, 2, \cdots$.
2. Let $f : \mathbb{R} \Rightarrow \mathbb{R}$ be a twice continuously differentiable function, with $f(0) = f(1) = f'(0) = 0$. Then **NET(Dec.): 2015**
(a) f'' is the zero function (b) $f''(0)$ is zero
(c) $f''(x) = 0$ for some $x \in (0, 1)$ (d) f'' never vanishes.
Ans. (c)
Hint. Please see the section 1.3.1.
3. Let $y(x)$ be the solution of the initial value problem $x^2y'' + xy' + y = x,\ y(1) = y'(1) = 1$, then the value of $y(e^{\frac{\pi}{2}})$ is **GATE(MA)-10**
A) $\frac{1}{2}(1 - e^{\frac{\pi}{2}})$ B) $\frac{1}{2}(1 + e^{\frac{\pi}{2}})$ C) $\frac{1}{2} + \frac{\pi}{4}$ D) $\frac{1}{2} - \frac{\pi}{4}$.
Ans. B)
Hint. Taking $x = e^z$ and the solution is $y(x) = \frac{1}{2}\cos(\log x) + \frac{1}{2}\sin(\log x) + e^x,\ x > 0$.
4. Let $y(x)$ be the solutions of the differential equation, $\frac{d}{dx}\left(x\frac{dy}{dx}\right) = x,\ y(1) = 0, \left(\frac{dy}{dx}\right)_{x=1} = 0$. Then $y(2)$ is **JAM(MA)-2016**
A) $\frac{3}{2} + \frac{1}{2}\ln 2$ B) $\frac{3}{2} - \frac{1}{2}\ln 2$ C) $\frac{3}{2} + \ln 2$ D) $\frac{3}{2} - \ln 2$.
Ans. B)
5. The sturm-Liouville problem $y'' + (\lambda)^2y = 0, y'(0) = 0, y'(\pi) = 0$ has its eigenvectors given by $y =$
(a) $\sin(n + \frac{1}{2})x$ (b) $\sin nx$ (c) $\cos(n + \frac{1}{2})x$ (d) $\cos nx$ **Gate(MA): 2000**
Ans. (d) is correct.
Hint: Where $\lambda = 0$ the solution is trivial. The solution of given differential equation is, $y = c_1 \cos \lambda x + c_2 \sin \lambda x$. Now $y' = \lambda(-c_1 \sin \lambda x + c_2 \cos \lambda x)$. Now $y'(0) = 0$ we get, $c_2 = 0$, and $c_1 \sin \lambda\pi = 0$. For $c_1 = 0$, solution is trivial. Now let $c_1 \neq 0$ then $\sin(\lambda\pi) = 0$ or, $\lambda\pi = n\pi,\ n \in Z$ or $\lambda = n$, thus $\lambda_n = n$. In other words λ_n be equal to one of the number $0, 1, 2, \cdots$. The eigenfunction is, $y_n = A_n \cos nx$.
6. Let $y(x)$ be the solution of the initial value problem
$$y''' - y'' + 4y' - 4y = 0,\ y(0) = y'(0) = 2,\ y''(0) = 0$$
then the value of $y(\frac{\pi}{2})$ is, **GATE(MA)-10**
A) $\frac{1}{5}(4e^{\frac{\pi}{2}} - 6)$ B) $\frac{1}{5}(6e^{\frac{\pi}{2}} - 4)$ C) $\frac{1}{5}(8e^{\frac{\pi}{2}} - 2)$ D) $\frac{1}{5}(8e^{\frac{\pi}{2}} + 2)$.

Ans. C)
Hint. The solution is $y(x) = \frac{8}{5}e^x + (\frac{2}{5}\cos 2x + \frac{1}{5}\sin 2x)$.

7. The solution of the differential equation $\frac{d^2y}{dx^2} - y = e^x$ satisfying the boundary conditions $y(0) = 0$ and $\frac{dy}{dx}(0) = \frac{3}{2}$ is
(a)$y(x) = \sinh x + \frac{x}{2}e^x$ (b)$y(x) = \sinh x - \frac{x}{2}e^x$
(c)$y(x) = \cosh x + \frac{x}{2}e^x$ (d)$y(x) = x\cosh x + \frac{x}{2}e^x$ [JAM CA-2010]
Ans. (a)

8. The solution to the initial value problem

$$\frac{d^2y}{dt^2} + \frac{dy}{dt} + 5y = 3e^{-t}\sin t,\ y(0) = 0,\ \left(\frac{dy}{dt}\right)_{x=0} = 2$$

is **GATE(MA)-14**
A) $y(t) = e^t(\sin t + \sin 2t)$ B) $y(t) = e^{-t}(\sin t + \sin 2t)$ C) $y(t) = 3e^t \sin t$ D) $y(t) = 3e^{-t}\sin t$.
Ans. B)

9. Consider the differential equation $y'' + 6y' + 25y = 0$ with initial condition $y(0) = 0$. Then the general solution of the IVP is
(a)$e^{-3x}(A\cos 4x + B\sin 4x)$ (b)$Be^{-3x}\sin 4x$
(c)$e^{-3x}(A\cos 4x + B\sin 3x)$ (d)$e^{-3x}(A\cos 3x + B\sin 3x)$ [JAM GP-2006]
Ans. (b)

10. The differential equation $y'' + y = 0$ satisfying $y(0) = 1$ and $y(\pi) = 0$ has
(a) a unique solution (b) a single infinite family of solutions
(c) no solution (d)A double infinity family of solutions [JAM GP-2008]
Ans. (b)

11. The solution of the differential equation $y'' + 4y = 0$ subject to $y(0) = 1$, $y'(0) = 2$ is
(a) $\sin 2x + 2\cos 2x$ (b) $\sin 2x - \cos 2x$ (c) $\sin 2x + \cos 2x$ (d) $\sin 2x + 2x$
Ans. (c) [JAM CA-2005]

12. The solution of the boundary value problem
$y'' + y = cosecx,\ 0 < x < \frac{\pi}{2},\ y(0) = 0,\ y(\frac{\pi}{2}) = 0$ is **NET(MS): (June)2012**
(a) convex (b) concave (c) negative (d) positive
Ans. (b) and (c)
Hint. $y(x) = A\cos x + B\sin x - x\cos x + \sin x \log|\sin x|$. Using boundary condition we get $y(x) = -x\cos x + \sin x\log|\sin x|$ so on $0 < x < \frac{\pi}{2}, y\frac{d^2y}{dx^2} < 0$ is convex condition of a function $y = f(x)$ and also $y < 0$. Hence (b) and (c) are corrects.

13. Let V be the set of all bounded solutions of the ODE
$u''(t) - 4u'(t) + 3u(t) = 0,\ t \in \Re$, Then V **NET(MS): (June)2012**
(a) is a real vector space of dimension 2 (b) is a real vector space of dimension 1
(c) contains only the trivial solution $u = 0$ (d) contains exactly two solution
Ans. (c)
Hint. $u(t) = Ae^{3t} + Be^t$. Since $u(t)$ is bounded for all t. As $t \to \infty$, $u(t)$ is bounded. Hence $A = B = 0$. Therefore, $u = 0$ is the only solution for this problem.

14. Let V be the set of all solution of the equation $y'' + ay' + by = 0$ satisfying $y(0) = y(1)$, where a, b are positive real numbers. Then the dimension(V) is equal to **GATE(MA): 2016**
(a) 2 (b) 1 (c) 0 (d) 3.

Ans. (b)
Hint. Here $V = \{Ay_1(x)+By_2(x) : 0 \le x \le 1\}$. Using boundary condition, we get, $A = f(B)$. Hence V contains only one arbitrary constant either A or B. So dimension(V)=1.

15. The boundary value problem $y'' + \lambda y = 0$ satisfying $y(-\pi) = y(\pi)$ and $y'(-\pi) = y'(\pi)$ to each eigenvalue λ, there corresponds **NET(MS): (June)2011**
(a) only one eigenfunction (b) two eigenfunctions
(c) two linearly independent eigenfunctions (d) two orthogonal eigenfunctions
Ans. (b), (c) and (d).
Hint. The eigenvalues problem is not a Sturm Liouville type because the two endpoint conditions are not "separated" between the two endpoints. Hence $\lambda_0 = 0$ is an eigenvalue with associated eigenfunction $y_0(x) \equiv 1$. Also there is no negative eigenvalues. Moreover, the $n-th$ position eigenvalue is n^2 and it has two linearly independent associated eigenfunctions $\cos nx$, $\sin nx$.

16. For the Sturm Liouville problems
$$(1 + x^2)y'' + 2xy' + \lambda x^2 y = 0$$
with $y'(1) = 0$ and $y'(10) = 0$ the eigenvalues, λ, satisfy **GATE(MA)-03**
A) $\lambda \ge 0$ B) $\lambda < 0$ C) $\lambda \ne 0$ D) $\lambda \le 0$.
Ans. A)

17. Let $k, l \in \mathbb{R}$ be such that every solution of $(D^2 + 2kD + l)y = 0$ satisfy $\lim_{x\to\infty} y(x) = 0$. Then **JAM(MA)-2017**
A) $3k^2 + l < 0$ and $k > 0$ B) $k^2 + l > 0$ and $k < 0$
C) $k^2 - l \le 0$ and $k > 0$ D) $k^2 - l > 0$, $k > 0$ and $l > 0$.
Ans. D)

18. Let y be a non-trivial solution of the boundary value problem $y'' + xy = 0$, $x \in [a, b]$ and $y(a) = y(b) = 0$ and then,
(a) $b > 0$ (b) y is monotone in $(a, 0)$ if $a < 0 < b$. **NET(MS)(Dec.)-2013**
(c) $y'(a) = 0$ (d) y has infinite many zeroes in $[a, b]$. **Ans.** (a) and (b).

19. Let $y(x)$ be the solution of the differential equation $\frac{dy}{dx} = (y - 1)(y - 3)$ satisfying the condition $y(0) = 2$. Then which of the following is TRUE ? **JAM(MA)-2017**
A) The function $y(x)$ is not bounded above. (B) The function $y(x)$ is bounded.
(C) $\lim_{x\to\infty} y(x) = 1$ D) $\lim_{x\to\infty} y(x) = 3$.
Ans. C)

20. The set of all eigenvalues of
$$y'' + \lambda y = 0, \ y'(0) = 0, \ y'(\frac{\pi}{2}) = 0$$
is **GATE(MA)-04**
A) $\lambda = 2n, \ n = 1, 2, 3, \cdots$ B) $\lambda = 4n^2, \ n = 1, 2, 3, \cdots$
C) $\lambda = n, \ n = 0, 1, 2, 3, \cdots$ D) $\lambda = 4n^2, \ n = 0, 1, 2, 3, \cdots$.
Ans. D)

21. Consider the BVP $u''(x) + \pi^2 u(x) = 0$, $x \in (0, 1)$, $u(0) = u(1) = 0$. If u and u' are continuous on $[0, 1]$, then **NET(MS): (June)2014**
(a) $\int_0^1 u'^3(x)dx = 0$ (b) $u'^2(x) + \pi^2 u^2(x) = u'^2(0)$

(c) $u'^2(x) + \pi^2 u^2(x) = u'^2(1)$. (d) $\int_0^1 u'^2(x)dx = \pi^2 \int_0^1 u^2(x)dx$

Ans. (b), (c), (d).

Hint. $u(x) = A\cos\pi x + B\sin\pi x$. Using boundary conditions, we get $A = 0$. Therefore $u(x) = B\sin\pi x \Rightarrow u'(x) = B\pi\cos\pi x$. Hence the results.

22. The eigenvalues of the boundary value problem $x'' + \lambda x = 0$, subject to the boundary conditions $x(0) = 0$, $x(\pi) + x'(\pi) = 0$ satisfy **[GATE 2000]**
(a) $\lambda + \tan\lambda\pi$ (b) $\sqrt{\lambda} + \tan\lambda\pi$ (c) $\sqrt{\lambda} + \tan\sqrt{\lambda}\pi$ (d) $\lambda + \tan\sqrt{\lambda}\pi$
Ans. (d).

23. The boundary value problem $x^2y'' - 2xy' + 2y = 0$, subject to the boundary conditions $y(1) + \alpha y'(1) = 1, y(2) + \beta y'(2) = 2$ has a unique solution if **[NET-DEC-2016]**
(a) $\alpha = -1$, $\beta = 2$. (b) $\alpha = -1$, $\beta = -2$. (c) $\alpha = -2$, $\beta = 2$. (d) $\alpha = -3$, $\beta = \frac{2}{3}$.
Ans: (a)
Hint. $y_1 = x$, $y_2 = x^2$ are two independent solutions. Using Theorem 1.4, we have $\Delta \neq 0$.

24. If $y = 3e^{2x} + e^{-2x}$ is the solutions of the initial value problem $\frac{d^2y}{dx^2} + \beta y = 4\alpha x$ with $y(0) = 4$ and $y'(0) = 1$ where $\alpha, \beta \in \mathbb{R}$, then GATE(MA)-2017
(a) $\alpha = 3, \beta = 4$ (b) $\alpha = 1, \beta = 2$ (c) $\alpha = 3, \beta = -4$ (d) $\alpha = 1, \beta = -2$
Ans. c.

1.10 Review Exercises

1 (a) Let ϕ_n be any function satisfying the boundary value problem

$$y'' + n^2 y = 0, \; y(0) = y(2\pi), \; y'(0) = y'(2\pi), \tag{1.79}$$

where $n = 0, 1, 2, \cdots$. Then show that $\int_0^{2\pi} \phi_n(x)\phi_m(x)dx = 0$, if $n \neq m$.

Hint. $-\phi_n'' = n^2\phi_n$, and $-\phi_m'' = m^2\phi_m$. Thus $(n^2 - m^2)\phi_m\phi_n = \phi_n\phi_m'' - \phi_m\phi_n'' = [\phi_n\phi_m' - \phi_m\phi_n']'$. Integrating this equality from 0 to 2π, and use the boundary conditions satisfied by ϕ_n and ϕ_m.

(b) Show that $\cos nx$ and $\sin nx$ are functions satisfying the boundary value problem (1.79). The result of (a) then implies that

$$\int_0^{2\pi} \cos nx \sin mx \, dx = 0, \; \int_0^{2\pi} \cos nx \cos mx \, dx = 0, \; \int_0^{2\pi} \sin nx \sin mx \, dx = 0, \; (n \neq m).$$

2 Show that $\phi_n(x) = \sin nx$ satisfies the boundary value problem $y'' + n^2 y = 0$, $y(0) = 0$, $y(\pi) = 0$, $n = 1, 2, \cdots$. Then show that $\int_0^{\pi} \cos nx \sin mx \, dx = 0$, $(n \neq m)$.

3 Let f be a real-valued continuous function on the strip

$$S : |x| \leq a, \quad |y| < \infty, \quad (a > 0),$$

and suppose f satisfies a Lipschitz condition on S. Show that the solution of the initial value problem $y'' + \lambda^2 y = f(x, y)$, $y(0) = 0$, $y'(0) = 1$, $(\lambda > 0)$, is unique.

4 Let f be a continuous function on an interval

$$|x - x_0| \leq a, \ (a > 0).$$

Show that the solution ϕ of the initial value problem $y'' = f(x), \ y(x_0) = \alpha, \ y'(x_0) = \beta$, can be written as $\phi(x) = \alpha + \beta(x - x_0) + \int_{x_0}^{x} (x - t) f(t)\, dt$.

5 Consider the special case

$$y'' + \sin y = 0,$$

which is an equation association with the oscillations of a pendulum. If ϕ is a solution satisfying

$$\phi(0) = 0, \ \phi'(0) = \beta > 0,$$

show that ϕ satisfies the equation

$$y' = \beta \sqrt{1 - k^2 \sin^2(\frac{y^2}{2})}, \ k = \frac{2}{\beta}.$$

6 Consider the constant coefficient equation $y'' + a_1 y' + a_2 y = 0$.
Let ϕ_1 be the solution satisfying

$$\phi_1(x_0) = 1, \quad \phi_1^{'}(x_0) = 0,$$

and let ϕ_2 be the solution satisfying

$$\phi_2(x_0) = 0, \quad \phi_2^{'}(x_0) = 1.$$

If ϕ is a solution satisfying

$$\phi(x_0) = \alpha, \quad \phi^{'}(x_0) = \beta,$$

show that

$$\phi(x) = \alpha\phi_1(x) + \beta\phi_2(x)$$

for all x.

7 Solve the initial value problem $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + 4y = 2x^2, \ x > 0$ given that $y(1) = 1, \ \frac{dy(0)}{dx} = 0$
[**Ans.** $y = (1 - 3\log x)x^2 + x^2(\log x)^2, \ x > 0$]

8 Let $y = f(x)$ be a twice continuously differential function on $(0, \infty)$ satisfying
$f(1) = 1$, and $f'(x) = \frac{1}{2}f(\frac{1}{x}), x > 0$, Solve the problem. JAM(MA)-2006
Ans. $f(x) = \sqrt{x}$
Hint. $f''(x) = -\frac{1}{x^2}\frac{1}{2}f'(\frac{1}{x}) = -\frac{1}{4x^2}f(x), \ f(1) = 1, \ f'(1) = \frac{1}{2}$.

9 Solve the initial value problem
$\frac{d^2y}{dx^2} - y = x(\sin x + e^x), \ y(0) = 0, \ y'(0) = 1$ JAM(MA)-2005
Ans. $y = \frac{5}{4}e^x + \frac{1}{4}e^{-x} - \frac{1}{2}x\sin x - \frac{1}{2}\cos x + \frac{1}{4}x^2e^x$

10 Solve the initial value problem $\frac{d^2y}{dx^2} + 4y = \sin 2x$ given that $y(0) = 1, \ \frac{dy(0)}{dx} = 0$ [**Ans.** $y(x) = \cos 2x + \frac{1}{8}\sin 2x - \frac{1}{4}x\cos 2x$]

11 Solve the initial value problem $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x$ given that $y(0) = 1, \ \frac{dy(0)}{dx} = 0$ [**Ans.** $y = (1 - x)e^x + \frac{1}{6}x^3e^x$]

12 Find the eigenvalues and corresponding eigenfunctions of the eigenvalues problem

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right)+\frac{\lambda}{x}y=0,\ (\lambda>0)$$

satisfying the boundary conditions $y'(1)=0$ and $y'(e^{2\pi})=0$.
Ans. $\lambda_n=\frac{n^2}{4},\ \phi_n(x)=A_n\cos\left(\frac{n}{2}\log x\right),\ n=1,2,3,\cdots,\ 1<x<e^{2\pi}$.

13 Find the eigenvalues and corresponding eigenfunctions of the eigenvalues problem

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right)+\frac{\lambda}{x}y=0,\ (\lambda>0)$$

satisfying the boundary conditions $y'(1)=0$ and $y'(e^{\pi})=0$.
Ans. $\lambda_n=n^2,\ \phi_n(x)=A_n\sin\left(n\log x\right),\ n=1,2,3,\cdots,\ 1<x<e^{\pi}$.

14 Find the eigenvalues and corresponding eigenfunctions of the eigenvalues problem

$$\frac{d^2y}{dx^2}+\lambda y=0,\ (\lambda>0)$$

with the following conditions
(a) $y(0)+y'(0)=0$ and $y(1)+y'(1)=0$. [C.U. (Hons.)-2002]
Ans. $\lambda_n=n^2\pi^2,\ \phi_n(x)=A_n\left(\sin n\pi x-n\pi\cos n\pi x\right),\ n=1,2,3,\cdots,\ 0<x<1$.
Hint. See the Example 1.17.
(b) $y(0)=0$ and $y(2\pi)=0$. [C.U. (Hons.)-1990, 2004]
Ans. $\lambda_n=\frac{n^2}{4},\ \phi_n(x)=A_n\sin\frac{nx}{2},\ n=1,2,3,\cdots,\ 0<x<2\pi$.

15 Consider the eigenvalue problem $y''+2y'+\lambda y=0;\ \ y(0)=y(1)=0$. Show that $\lambda=1$ is not an eigenvalue. (b) Show that there is no eigenvalue λ such that $\lambda<1$. (c) Show that the $n-th$ positive eigenvalue is $\lambda_n=n^2\pi^2+1$, with associated eigenfunction $y_n(x)=e^{-x}\sin n\pi x$.

16 Put, if possible, the following equations in Sturm-Liouville form
(a) $\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+(x+\lambda)y=0$ [**Ans.** $\frac{d}{dx}\left(e^{x^2}\frac{dy}{dx}\right)+xe^{x^2}y+\lambda e^{x^2}y=0$]
(b) $\frac{d^2y}{dx^2}+\lambda y=0$ [**Ans.** $\frac{d}{dx}\left(\frac{dy}{dx}\right)+\lambda y=0$]

17 Let $y(x)$ be the solution of the differential equation, $y''+5y'+6y=0,\ y(0)=1,\ \left(\frac{dy}{dx}\right)_{x=0}=-1$. Then $y(x)$ attains it maximum value at $x=?$ **JAM(MA)-2016**
Ans. $-0.3,\ -0.25$.

18 If $\frac{d^2y}{dx^2}+\lambda^2y=0,\ y(0)=y_0$ and $y(l)=0$, prove that $y(x)=\frac{y_0\sin\lambda(l-x)}{\sin\lambda l},\ 0\le x\le l$.

Bibliography

[1] Birkhoff, G. and Rota, G.C. (1989). Ordinary Differential Equations (4th ed.), Wiley, New York.

[2] Bernstein, D.L., (1951). Existence Theorems in Partial Differential Equations (AM-23), Annals of Mathematics Studies, Princeton University Press.

[3] Boyce, W. E. and DiPrima, R. C. (2001). Elementary Differential Equations and Boundary Value Problems, John Wiley & Sons, Inc. New York.

[4] Chatterjee, D., (2000). Integral Calculus and Differential Equations, Tata McGraw-Hill Publishing Company Limited, New Delhi.

[5] Coddington, E. A.(1961). An Introduction to Ordinary Differential Equations, Dover Publications, Inc., New York.

[6] Debnath, L. and Bhatta, D. (2006). Integral Transforms and Their Applications, Chapman & Hall/CRC Taylor & Francis Group.

[7] Jana, D. K. and Maity, K. (2013). Advance Engineering Numerical Methods, Macmillan Publisher India Ltd.

[8] Jordan, D. W. and Smith, P. (2009). Nonlinear Ordinary Differential Equations, Oxford University Press, Oxford.

[9] Jain, M. K., Iyengar, S.R.K. and Jain, R.K. (1984). Numerical Methods for Scientific and Engineering Computation, New Age International Publishers, New Delhi.

[10] Ghosh, R. K. and Maity, K. C.(2011). An Introduction to Differential Equations (9th Edition) New Central Book Agency (P) Ltd., Kolkata.

[11] Khatua, D. and Maity, K.(2016). "Stability Analysis of a Dynamical System", (edited by Maity, K.) in "Bio-mathematical Modeling under Uncertain Environment". NAROSA, New Delhi.

[12] Liao, X., Wang, L. and Yu, P. (2007). Stability of a dynamical system. Elsevier.

[13] Liu, W.M. (1994). Criterion of Hopf bifurcation without using eigenvalues. J. Math. Anal. Appl, 182 : 250.

[14] Ma, Z.,Wang, Y. and Jiang, G. (2012). Bifurcation analysis of a linear Hamiltonian system with two kinds of impulsive control. Nonlinear Dyn., 70 : 2367-2374.

[15] Maity, K. (2016). Bio-Mathematical Modelling Under Uncertain Environment(Edited Book), Narosa, New Delhi.

[16] Makinde, O.D. (2007). Solving ratio-dependent predatorprey system with constant effort harvesting using Adomain decomposition method. Appl. Math.Comput, 187 : 17-22.

[17] Mazandarani, M., Najariyan, M.,(2014). A note on "A class of linear differential dynamical systems with fuzzy initial condition", Fuzzy Sets and Systems, Article in press.

[18] Mondal, N.(2015). Differential Equations (Ordinary and Partial), Books and Applied (P) Ltd, Kolkata.

[19] Mondal, S. K.,(2016). "An Overview on Stability Analysis of a Nonlinear System and its Application in Prey-Predator Model", (edited by Maity, K.) in "Bio-Mathematical Modelling Under Uncertain Environment", Narosa, New Delhi.

[20] Morley, M. (1965). Categoricity in Power. Transactions of the American Mathematical Society (American Mathematical Society), 114 (2): 514-538.

[21] Mukherjee, A., Bej, N. K. (2009). Ordinary and Partial Differential Equations, Shreedhar Prakashani, Kolkata.

[22] Xie, W. C.(2010). Differential Equations for Engineers, Cambridge University Press, Cambridge.

[23] Parks, P.C. and Hahn, V.,(1992). Stability Theory, Prentics Hall, New York.

[24] Perko, L. (1991). Differential Equations and Dynamical Systems. Springer.

[25] Raisinghania, M,D. (2011). Ordinary and Partial Differential Equations, S. Chand and Company Ltd., New Delhi.

[26] Rao, K. S.(2009). Introduction to Partial Differential Equations, Phi Learning Private Limited, New Delhi.

[27] Ross, S. L.(2004). Differential Equations, John Wiley & Sons(Asia) Pre. Ltd., Singapore.

[28] Shelah, S. (1969). Stable theories. Israel J. Math., 7 (3): 187-202.

[29] Shelah, S. (1990). Classification theory and the number of nonisomorphic models. Studies in Logic and the Foundations of Mathematics (2nd ed.), Elsevier.

[30] Somasundaram, D. (2010). Ordinary Differential Equations-A first cource, Narosa Publishing House.

[31] Spiegel, M.R. (1965). Theory and Problems of Laplace Transforms, Schaum's Outline Series, MacGraw-Hill Book Company, Inc.

[32] Weinberger, H. (1995). A First Course in Partial Differential Equations, Wiley, New York.

[33] Xiao, D., Li, W. and Han, W. (2006). Dynamics in a ratio dependent predator prey modelwith predator harvesting. J. Math.Anal. Appl., 324(1): 4-29.

Index